RESEARCH STATEMENT - ERIK HOLMES

1. Introduction

My research lies in *algebraic number theory* and *arithmetic statistics*, with a strong emphasis on computational methods. Broadly, I study the geometry and the distribution of certain arithmetic invariants within natural families of number fields. This research involves the use of many fields and methods—such as algebra, analysis, geometry, and computation—and there are various points of entry which make it especially well-suited for collaborations and student-driven projects.

1.1. **Number theory.**
One of the joys of number theory is the accessibility of its questions, from the mystery of twin primes ("How many pairs of primes differ by two?") to the story of Fermat's Last Theorem inspiring a young Andrew Wiles, deep problems can often be stated in the simplest of terms. This accessibility can excite a general audience and provides a natural entry point for students to engage with mathematics early in their careers.

Equally compelling is the experimental side of modern number theory. Through computation, we can explore small cases, generate data, and detect patterns that guide conjectures and eventually proofs. This interplay between experiment and theory has become a central feature of my own research. It also creates rich opportunities for projects where students can learn programming, data analysis, and mathematical reasoning to contribute to research while also serving to motivate some of the technical background necessary to work on open questions.

1.2. **Arithmetic statistics and the geometry of number fields.**
Arithmetic statistics seeks to understand arithmetic objects by studying them in families and describing their asymptotic behavior. A hallmark question in the field, known as Goldfeld's conjecture, is *"what proportion of elliptic curves have rank 0 (rank 1)"*? Since Bhargava's groundbreaking work on the geometry of numbers and sieve theory, the field has undergone a renaissance.

My research program focuses on the Archimedean side of arithematic statistics, where the key objects are lattices attached to number fields and their shapes—geometry modulo scaling, rotation, and reflection. In some families shapes appear random; in others, they are rigidly confined to subspaces. My recent work shows that both additive and multiplicative lattices exhibit such rigidity, and that these constraints can sometimes explain subtle phenomena in field-counting asymptotics, such as the logarithmic deviations underlying counterexamples to Malle's conjecture (see Section 3.3 and Section 5.1).

1.3. **Motivation and broader impacts.**
What excites me most is that this program is both deep and accessible. Shapes can be explored computationally, making it easy to generate examples, spot patterns, and pose conjectures—activities that naturally engage students. At the same time, these explorations connect to major themes in number theory, cryptography, and beyond. My long-term goal is to continue developing a research program that uncovers the principles governing rigidity in shapes, connects them to dynamics on homogeneous spaces, and explores the implications to lattice-based cryptography, where lattice geometry is central to post-quantum security. These avenues support meaningful student projects that integrate proof, computation, and data analysis—developing a breadth of methods and tools valuable for both mathematical research and industry careers.

2. Shapes of lattices in number theory

Lattices play a central role across mathematics—from Minkowski's geometry of numbers and the finiteness of class groups, to sphere packing and Viazovska's breakthrough, to lattice-based cryptography. In each case, understanding the geometry of the relevant lattices is crucial.

Formally, if Λ is a lattice in a real inner product space, we define the *shape* of Λ to be its equivalence class under scaling, rotation, and reflection. The space of shapes (of rank n lattices) can be realized as the double coset space:

$$\mathcal{S}_n := \mathrm{GL}_n(\mathbf{Z})\backslash\mathrm{GL}_n(\mathbf{R})/\mathrm{GO}_n(\mathbf{R}),$$

where $\mathrm{GO}_n(\mathbb{R}) = \{g \in \mathrm{GL}_n(\mathbb{R}) : g^\top g = \lambda I \text{ for some } \lambda > 0\}$. This quotient inherits a natural Haar measure, which can be normalized to a probability measure on a fundamental domain. Our goal is to study natural families of lattices and the distribution of their shapes with respect to this measure (or its restrictions to relevant subspaces).

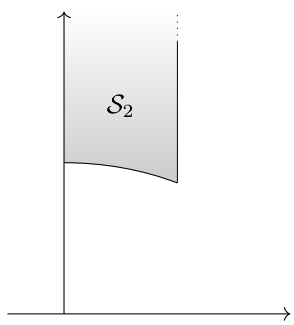


Figure 1. Space of shapes of rank 2 lattices

Example 2.1. *When $n = 2$ it is convenient to identify $\mathbf{R}^2 \cong \mathbf{C}$. Any rank-2 lattice has a basis $\Lambda = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ with $\tau := \omega_2/\omega_1 \in \mathfrak{h}$ (the upper half-plane). Multiplying (ω_1, ω_2) be a nonzero complex scalar (scaling/rotation) does not change τ; reflection sends $\tau \to -\bar{\tau}$. Changing the $\mathbf{Z}$-basis by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ sends*

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

Thus the shape (i.e. the lattice up to scaling, rotation, and reflection) corresponds to the orbit of τ under $\mathrm{GL}_2(\mathbf{Z})$, and the space of shapes of rank-2 lattices is the modular curve $\mathrm{GL}_2(\mathbf{Z})\backslash\mathfrak{h}$. A fundamental domain is

$$\left\{ z \in \mathfrak{h} : |Re(z)| \leq \frac{1}{2}, |z| \geq 1 \right\}$$

as seen in Figure 1. Note, the special (cusp) points i (square lattice) and $\frac{1+i\sqrt{3}}{2}$ (hexagonal lattice).

2.1. **Lattices in number theory.** A central object in number theory is a number field, a finite extension of $\mathbf{Q}$, and its ring of integers $\mathcal{O}_K$. Let $K/\mathbf{Q}$ be a number field of degree $n = r + 2s$ with r real embeddings $\sigma_1, \ldots, \sigma_r$ and s pairs of complex embeddings $\{\tau_1, \overline{\tau}_1\}, \ldots, \{\tau_s, \overline{\tau}_s\}$. Minkowski's *geometry of numbers* embeds $\mathcal{O}_K$ as a lattice inside the real vector space

$$K_{\mathbf{R}} \;:=\; K \otimes_{\mathbf{Q}} \mathbf{R} \;\cong\; \mathbf{R}^r \times \mathbf{C}^s \;\cong\; \mathbf{R}^{r+2s}.$$

The Minkowski embedding

$$j : K \longrightarrow \mathbf{R}^r \times \mathbf{C}^s, \qquad \alpha \longmapsto \big(\sigma_1(\alpha), \ldots, \sigma_r(\alpha),\ \tau_1(\alpha), \ldots, \tau_s(\alpha)\big),$$

is $\mathbf{Q}$-linear, and $j(\mathcal{O}_K)$ is a full-rank lattice in $K_{\mathbf{R}}$ (e.g., for the inner product induced by the trace form). Equivalently, identifying $\mathbf{C} \cong \mathbf{R}^2$, one may view ι as an embedding into $\mathbf{R}^{r+2s}$ (often with a $\sqrt{2}$-factor on the complex coordinates to make it an isometry).

Alongside this *integral lattice*, there is a multiplicative analogue coming from the unit group. Let $\mathcal{O}_K^\times$ be the units of K and $\mu(K)$ be the roots of unity. The logarithmic embedding maps the units of K into $\mathbf{R}^{r+s}$:

$$\mathrm{Log} :\ \mathcal{O}_K^\times \longrightarrow \mathbf{R}^{r+s}, \qquad \alpha \longmapsto \big(\log|\sigma_1(\alpha)|, \ldots, \log|\sigma_r(\alpha)|,\ 2\log|\tau_1(\alpha)|, \ldots, 2\log|\tau_s(\alpha)|\big).$$

Its image lies in the trace-zero hyperplane $\{(x_1, \ldots, x_{r+s}) : \sum_i x_i = 0\}$ by the product formula, and Dirichlet's unit theorem asserts that $\Lambda_K := \mathrm{Log}(\mathcal{O}_K^\times/\mu(K))$ is a lattice of rank $r + s - 1$ in this hyperplane.

These two lattices $j(\mathcal{O}_K)$ and Λ_K are the main players in my research program. In the next section we formalize both the families that we investigate and the questions—existence, constraints, and distribution of shapes—that we pursue.

2.2. **Imposing Galois conditions.** A standard way to organize families of number fields is by Galois group. For a (not necessarily Galois) number field $K/\mathbf{Q}$, we say that K has Galois group G if its Galois closure $\widetilde{K}/\mathbf{Q}$ satisfies $\mathrm{Gal}(\widetilde{K}/\mathbf{Q}) \cong G$. Fix a signature $n = r + 2s$ and write

$$\mathcal{F}(G, r, s) \;:=\; \{\, K/\mathbf{Q} :\ [K : \mathbf{Q}] = n,\ \mathrm{Gal}(\widetilde{K}/\mathbf{Q}) \cong G,\ \mathrm{sig}(K) = (r, s) \,\}.$$

We then study the image of this family in the space of shapes:

$$\Omega_?(G, r, s) \;:=\; \big\{ \mathrm{sh}_?(K) \in \mathcal{S}_{n-1} :\ K \in \mathcal{F}(G, r, s) \big\}.$$

where $? = \{+, \times\}$ and $\mathrm{sh}_+(K)$ (resp. $\mathrm{sh}_\times(K)$) is the shape of the additive (resp. multiplicative) lattice of K. Three goals guide our program:

Q1. On which (typically proper) loci of $\mathcal{S}_{n-1}$ does $\Omega_?(G,r,s)$ lie? (E.g., unions of torus orbits, boundary strata, or other explicitly described subvarieties.)
Q2. How are shapes distributed on those loci? (E.g., equidistribution versus concentration, limiting measures, accumulation on special points.)
Q3. How powerful is the shape as an invariant of number fields? *(Additive vs. multiplicative.)* To what extent does the shape determine K? In particular, is the shape *(i)* a complete invariant on certain families (i.e., $\mathrm{sh}(K) = \mathrm{sh}(L) \Rightarrow K \cong L$), or at least *(ii)* finite-to-one?

We emphasize the contrast between the *generic* case $G = S_n$, where few constraints are expected, and *non-generic* cases $G \subsetneq S_n$, where strong geometric restrictions often appear. Beyond intrinsic interest, identifying these loci and distributions may inform asymptotic predictions for number fields (in the spirit of Malle-type conjectures).

2.3. **Lattices and cryptography.** Beyond their intrinsic interest in number theory, lattices play a central role in modern cryptography. Post-quantum 'lattice-based' schemes rely on the presumed hardness of geometric problems such as the Shortest/Closest Vector Problems (SVP/CVP) in high dimensions. Because hardness is sensitive to lattice shape and related invariants, arithmetic constraints on shapes can expose brittle instances or certify robust ones. This dovetails with my program: by analyzing how number-theoretic structure restricts the space of shapes, we can map which regions of the shape moduli are cryptographically fragile versus safe. I'm exploring collaborations to translate these rigidity results into guidance for concrete schemes and parameter choices.

3. The additive case: shapes of number fields

Given a degree n number field K, the Minkowski embedding $\iota : \mathcal{O}_K \hookrightarrow K_{\mathbf{R}}$ identifies $j(\mathcal{O}_K)$ with a rank-n lattice in $K_{\mathbf{R}} \cong \mathbf{R}^{r+2s}$. Because there is a copy of $\mathbf{Z}$ in every field $1 \in \mathcal{O}_K$ spans the distinguished line $L := \mathbf{R} \cdot j(1)$, and since we will study these lattices in families of number fields we instead pass to its orthogonal complement

$$H \;:=\; \{x \in K_{\mathbf{R}} :\; \langle x, j(1)\rangle = 0\},$$

the trace-zero hyperplane. Let

$$j(\mathcal{O}_K^{\perp}) \;:=\; \pi_H\big(j(\mathcal{O}_K)\big) \subset H$$

be the projection of $j(\mathcal{O}_K)$ to H; this is a full-rank lattice of rank $n-1$. We define the *shape* of K to be

$$\mathrm{sh}_+(K) \;:=\; [j(\mathcal{O}_K^{\perp})] \;\in\; \mathcal{S}_{n-1}.$$

3.1. **Background.** The first nontrivial case arises in degree 3. Terr [Ter97] proved that the shapes of (non-Galois) cubic fields are equidistributed in $\mathcal{S}_2$ with respect to the $\mathrm{GL}_2(\mathbf{R})$-invariant measure. By contrast, in the Galois (cyclic cubic) case the extra symmetry collapses all shapes to a single $\mathrm{GL}_2(\mathbf{Z})$-orbit—namely the hexagonal lattice on the boundary. Thus, in the "generic" case there are no geometric constraints (equidistribution in $\mathcal{S}_2$), whereas in the non-generic C_3 case the symmetry imposes extreme rigidity: every such field has the same shape.

Building on Bhargava's parametrizations, [Bha04b, Bha08], Bhargava–Harris[1] [BH16] extended equidistribution to S_n-fields for $n = 4, 5$. Beyond such generic families far less is known: work of Harron [Har17, Har19], Harron–Harris [HH20], Hassan [Has19], and others shows that in many non-generic settings (e.g., cyclic, biquadratic, multiquadratic) the Galois structure constrains shapes to a proper locus (e.g., unions of torus orbits or boundary strata) *and* that, within this locus, the shapes are proved or conjectured to be equidistributed with respect to the measure induced from the $\mathrm{GL}_n(\mathbf{R})$-invariant measure on $\mathcal{S}_{n-1}$.

3.2. **Results.** One recent project of mine focuses on *pure prime degree fields*, $K = \mathbb{Q}(\sqrt[p]{m})$. Harron, in [Har17], addressed questions Q1, Q2, and Q3 in the case of pure cubic fields showing that the shapes are equidistributed along vertical geodesics in $\mathcal{S}_2$, and that the shape is a complete invariant. In my work [Hol22], I generalized this to all prime degrees p. Specifically, I showed that in the wildly ramified case the shape of K lies on an explicit torus orbit $\mathscr{S}_p \subset \mathcal{S}_{p-1}$:

For a prime p (think $K = \mathbf{Q}(\sqrt[p]{m})$), we looked at the *shape* $\mathrm{sh}_+(K)$ of the additive lattice coming from $\mathcal{O}_K$.

[1] In this note Harris refers to Piper Harris, who has published under Piper Harron, and Piper H

Theorem 3.1 ([Hol22])**.** *For every (wildly ramified) pure prime–degree field K, the shape* $\mathrm{sh}(K)$ *lies on a single one–dimensional orbit $\mathscr{S}_p$ cut out by the diagonal torus. In short, all such shapes live on the same explicit curve $\mathscr{S}_p \subset \S_{p-1}$.*

This curve $\mathscr{S}_p$ carries a natural "length" measure μ_p, coming from Haar measure on the torus. With respect to this canonical choice, we show:

Theorem 3.2 ([Hol22])**.** *As you range over pure prime–degree fields, the shapes are equidistributed along $\mathscr{S}_p$ with respect to μ_p.*

While equidistribution of shapes had previously been proved only in a few low-degree cases, this paper establishes it in an infinite number of prime degrees. As a result, we have ample examples of lattices (with arbitrarily high rank) that could serve as a natural testing ground to explore how shape information might influence lattice-based cryptography.
A concrete consequence in the quintic case ($p = 5$). If $W \subset \S_4$ is any reasonable set, and $N_5(X, W)$ counts pure quintic fields K with $|\Delta(K)| < X$ and $\mathrm{sh}(K) \in W$, then

$$N_5(X, W) \ \sim\ C\, X^{1/4} \log(X) \cdot \mu_5(W),$$

for an explicit constant $C > 0$. In words: the overall growth is $X^{1/4} \log X$, and the fraction falling in W is exactly its μ_5-length inside the shape–curve $\mathscr{S}_5$. Finally, we addressed Q3 by proving:

Theorem 3.3 ([Hol22])**.** *Within the family of pure prime–degree number fields $K = \mathbf{Q}(\sqrt[p]{m})$ (with p prime), the shape is a* complete invariant*:*

$$\mathrm{sh}_+(K) = \mathrm{sh}_+(L) \quad \Longleftrightarrow \quad K \cong L.$$

3.3. **Ongoing projects.** In joint work with Harron and Varma [HHV26], we are investigating the first natural family in which Malle's conjecture is known to fail: non-Galois sextic fields whose Galois closures have Galois group $C_3 \wr C_2$ (the transitive imprimitive subgroup of S_6 of order 18).

3.3.1. *Malle's conjecture.* The Hermite–Minkowski theorem implies that, for any $X > 0$, there are only finitely many number fields with $|\Delta(K)| \leq X$. Malle [Mal02] conjectured precise asymptotics for the counting function

$$N_{n,G}(X) \ :=\ \#\{\, K/\mathbf{Q} :\ [K : \mathbf{Q}] = n,\ \mathrm{Gal}(\widetilde{K}/\mathbf{Q}) \cong G,\ |\Delta(K)| < X \,\}.$$

Klüners [Klü05] produced a counterexample in the sextic case $G = C_3 \wr C_2$, showing that $N_{6,G}(X)$ acquires an additional logarithmic factor beyond Malle's prediction. Strikingly, this extra $\log X$ term occurs precisely when the quadratic resolvent (equivalently, the distinguished quadratic subfield of the Galois closure) is cyclotomic, $\mathbf{Q}(\zeta_3) = \mathbf{Q}(\sqrt{-3})$, indicating that hidden geometric/arithmetical structure is at play within this family.

3.3.2. *A geometric interpretation via shapes.* Our approach was to ask: *can the geometry of lattice shapes detect this difference?* To answer this, we refined the notion of shape. Given a sextic field K with quadratic subfield F, we define the *F-shape* of K by projecting the integral lattice to the orthogonal complement of $j(\mathcal{O}_F)$. This framework extends the usual notion of shape and allows one to study families with fixed subfields.

We prove two main results. First, for each quadratic subfield $F \subset K$, the F-shapes of these sextic fields lie on a one–dimensional subspace (a torus orbit)

$$\mathscr{S}_F \ \subseteq\ \mathcal{S}_4.$$

Second, this subspace has finite measure when $F \neq \mathbf{Q}(\zeta_3)$ but infinite measure when $F = \mathbf{Q}(\zeta_3)$. Consequently, the distribution of shapes along $\mathscr{S}_F$ matches the asymptotics discovered by Klüners: finite induced measure corresponds to Malle's predicted growth, whereas infinite induced measure yields the additional logarithmic factor.

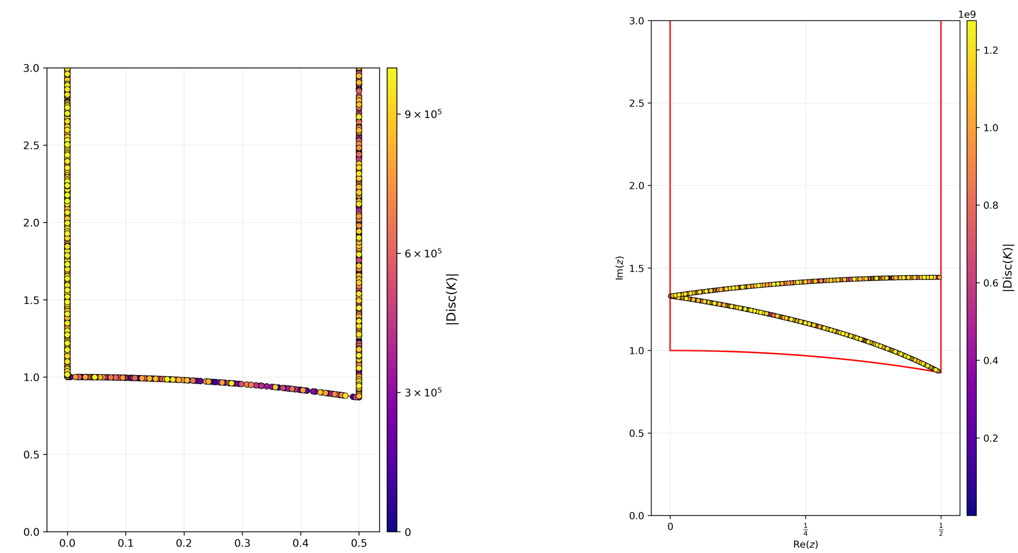
This shows that lattice geometry does not merely reflect known asymptotics but can actually *explain* subtle deviations. My current and future work pursues this perspective: extending the theory of F-shapes to broader families, understanding how subspaces of shape space govern field-counting asymptotics, and developing a general framework for predicting when rigidity in shapes forces anomalies in conjectural heuristics.

4. The multiplicative case: shapes of unit lattices

Alongside the classical Minkowski lattice sits the *log–unit lattice*, the image of $\mathcal{O}_K^\times/\mu(K)$ under the logarithmic embedding. It has rank $r+s-1$ (Dirichlet) and a natural shape in $\mathcal{S}_{r+s-1}$. Despite its simplicity to define, the geometry of unit shapes is subtle, and even basic classification and distribution questions remain challenging.

4.1. **Background.** David–Shapira [DS18] conjectured that the shapes of totally real cubic fields are dense in $\mathcal{S}_2$, a prediction recently proved (for nonmaximal orders) by Dang–Gargava–Li [DGL25]. In contrast, for several non-generic families the Galois module structure of units forces the shapes onto special loci. Recent work in the biquadratic and cyclic settings (e.g., [ATPS20, Cru20]) exhibits constraints such as orthogonality or well-roundedness, reflecting genuine geometric restrictions on the unit lattices. Taken together, these results mirror the additive case: in some families unit shapes behave "randomly" (e.g., are dense), while in others they are highly rigid and confined to explicit subspaces.

4.2. **Results.** My recent projects have focused primarily on question Q1 for log–unit lattices in dihedral families. Specifically, we identify the image $\Omega_{\log}(G,r,s)$ inside the ambient shape space and find striking rigidity phenomena:



((a)) Shapes of D_4 quartic fields lie on the boundary of $\mathcal{S}_2$ and appear to be equidistributed.

((b)) Shapes of D_5 quintic fields lie on a fixed hypercycle in the upper half-plane.

Figure 2. Rigidity phenomena for unit log lattices: in both D_4 and D_5 families the shapes are constrained to strict geometric subspaces.

- D_p **fields.** In joint work with Harron and Vemulapalli, we consider unit shapes of prime–degree dihedral number fields (where D_p is the symmetry group of a regular p-gon). We prove:

  Theorem 4.1 ([HHV25])**.** *The unit shapes of prime–degree dihedral fields lie on a finite union of explicit diagonal–torus orbits.*

  In the quintic case ($p=5$) the log–unit shape is constrained to a single *hypercycle* (the set of points at fixed hyperbolic distance from a geodesic); see the explicit curve in Figure 2(b).

- D_4 **quartic fields.** In recent joint work [CCH+25], we investigate quartic D_4 fields of signature $(2,1)$ and describe the locus of shapes:

  Theorem 4.2 ([CCH+25])**.** *The unit shapes are constrained to the* boundary *of* $\S_2$ *and have transcendental coordinate(s).*

Moreover, we determine whether the shape lies on the left, right, or bottom boundary in terms of norm maps to the quadratic subfield, and we exhibit boundary limit points via explicit families.

Together these results reveal a recurring phenomenon: additional Galois symmetry rigidifies the unit lattice, forcing shapes onto lower–dimensional loci—explicit torus orbits in the D_p case, and boundary strata in the D_4 case. In two dimensions these geometric constraints also reflect concrete lattice features, e.g., orthogonality (left boundary), the existence of a Minkowski basis (generated by a vector and its Galois conjugates) and whether such a basis is reduced, or the appearance of shorter basis vectors.

4.3. **Ongoing work.** In an ongoing collaboration involving graduate students we consider the case of totally imaginary D_6-sextic fields. We first addressed Q1, identifying the space of shapes within this family of fields, where we have discovered a dichotomy: if the cubic subfield is totally real (i.e. the sextic field is CM), then the unit shape lies in the interior of the modular curve; if the cubic subfield is complex, the shape is restricted to the boundary.

We then explore Q3, the extent to which the unit shape determines the field itself. Let $\mathcal{F}_r$ (resp. $\mathcal{F}_c$) denote the family of imaginary D_6-sextic fields with real (resp. complex) cubic subfield. We show that for every field in $\mathcal{F}_r$, the shape is inherited from its cubic subfield and therefore there are infinitely many fields with the same shape (taking any imaginary quadratic extension of the totally real field yields the same shape). In contrast, we also prove:

Theorem 4.3. *The unit shape is a complete invariant for* $\mathcal{F}_c$.

In other words, within the two subfamilies, the same invariant exhibits opposite behaviors—redundant in one case and complete in the other.

Furthermore, we consider the geometric properties of well roundedness and orthogonality, exhibiting an infinite family of well rounded lattices that-despite the coordinates being transcendental-converge to the hexagonal cusp. We also show that if the cubic subfield is pure (e.g. $\mathbf{Q}(\sqrt[3]{2})$), then *at least 49.8% of the unit lattices are orthogonal,* which provides quantitative evidence of orthogonality phenomena in higher-degree number fields.

5. Future work

In this section I outline ongoing and planned directions. A feature of this program is that each thread naturally generates problems at multiple levels, from computational experiments to theoretical questions.

5.1. **Shapes of number fields.** A broad goal is to make precise the connection between lattice shapes and number field asymptotics. Given a family $\mathcal{F}$ of number fields of fixed degree and Galois group, one would like to:

(a) determine the locus $\mathscr{S}_{\mathcal{F}} \subset \mathcal{S}_{n-1}$ cut out by the additive shapes $\Omega_{\text{add}}(\mathcal{F})$, and
(b) use the geometry and distribution of $\mathscr{S}_{\mathcal{F}}$ to inform the asymptotics of $\mathcal{F}$.

Work of Harron, Harris, Hassan, and myself in non-generic families provides first examples; many cases remain open.

In [Hol22] we also pose several questions I plan to pursue with students. For instance, building on [BS14], where pure cubic fields correspond to S_3-cubics with cyclotomic quadratic resolvent, we prove in [Hol22] that if K has Galois closure with group F_p (the Frobenius group of order $p(p-1)$), then K is pure iff its $(p-1)$-degree resolvent field is cyclotomic. This leads to:

Question. What are the shapes and distributions of F_p-fields whose resolvent field is *not* cyclotomic?

A plausible route is: (i) initial computation to map the attainable loci in $\mathcal{S}_{p-1}$; (ii) structural analysis of the resulting integral lattices and their rigid subspaces; (iii) parametrization of the family and a measure-theoretic study of distribution on the locus; and (iv) a comparison with field-counting asymptotics (Malle-type predictions), especially the effect of excluding the cyclotomic resolvent.

5.2. **Shapes of unit lattices and higher-rank investigations.** Another direction is to extend the study of log-unit shapes beyond the initial dihedral cases. Evidence to date indicates strong restrictions in higher rank, often dictated by the $\mathbf{Z}[G]$-module structure of $\mathcal{O}_K^\times/\mu(K)$. Current collaborations within my research groups aim to chart these loci and testing for rigidity phenomena in new signatures. This area remains underexplored and offers many student-friendly projects—from explicit computation and conjecture testing to proving new constraints.

5.2.1. *Connections to counting number fields.* Because unit shapes are comparatively new as invariants, they may yield additional traction on asymptotics. In ongoing work with Ila Varma, we are investigating whether D_5 unit shapes can be linked to Bhargava's parametrization of all rank 5 rings, with the goal of distinguishing D_5 rings inside this parametrization. Establishing such a link would be a step toward asymptotics for D_5 fields.

5.3. **Shapes of function fields.** In joint work with Bauer, Nguyen, Scheidler, and Tran, we investigate a function-field analogue of lattice shapes. For pure cubic function fields $K = \mathbb{F}_q\big(t, f(t)^{1/3}\big)$, we find: when $q \equiv 1 \pmod 3$ (Kummer applies) the shape is fixed; when $q \equiv -1 \pmod 3$ the shapes lie on a one-dimensional torus orbit, closely paralleling the number-field case. We are studying the distribution on this locus and expect a preprint soon. Broadly, these results indicate that methods and questions for number fields carry over to the function-field setting, with both new phenomena and powerful analogies.

5.4. **Summary.** Taken together, these projects advance a unified program: using lattice shapes to reveal hidden structure in number fields and, in turn, to inform asymptotics. The results to date—explicit loci, equidistribution on constrained supports, and complete invariants in natural families—open many directions that scale from computation to theory. I look forward to pursuing these questions with students and collaborators, building a steady pipeline of problems where experiments suggest conjectures and structure the proofs that follow.

6. Other directions

Beyond lattice geometry, I have pursued problems in elementary number theory and arithmetic dynamics. Negative Pell over function fields. While at Calgary I co-supervised the M.Sc. thesis of Fujikawa [Fuj23] on the negative Pell equation over function fields, aiming toward a function-field analogue of Stevenhagen's conjecture. The result of this thesis was not a full analogue of Stevenhagen's conjecture but a necessary first step in this direction. Future work, which is well suited for students, would be to extend this work and establish the analogous conjecture in positive characteristic: this would involve designing computational experiments, isolating the governing local/global conditions, and push toward a full analogue of Stevenhagen's prediction.

Adelic perturbations of rational functions. In recent work with Baril Boudreau and Nguyen [BHN25], we study power series of the form *adelic perturbations* of rational functions, motivated by questions on dynamical (Artin–Mazur) zeta functions. We prove a strong Pólya–Carlson dichotomy under a mild stability hypothesis via a framework of *almost quasi-polynomials* that unifies p-adic and Archimedean techniques. Specifically:

Theorem 6.1 ([BHN25])**.** *Let $\sum_{n\geq 0} a_n x^n \in \bar{\mathbf{Q}}[[x]]$ be the power series of a rational function (radius of convergence R), and let $f : \mathbb{N}_0 \to \bar{\mathbf{Q}}$ be an almost quasi-polynomial. Then the perturbed series*

$$\sum_{n\geq 0} f(n)\, a_n\, x^n$$

satisfies the Pólya–Carlson dichotomy: it is either rational or admits the circle of radius R as a natural boundary.

This settled several open problems, including: a conjecture of Byszewski–Cornelissen on analytic continuation of Artin–Mazur zeta functions for endomorphisms of abelian varieties, a question of Bell–Miles–Ward on zeta functions of algebraic dynamical systems, and a conjecture of Royals–Ward on adelic perturbations. More broadly, the method provides a systematic route to analytic dichotomy phenomena in arithmetic/algebraic dynamics and opens a range of follow-up problems suitable for student involvement.

References

[ASVW19] A. Altug, A. Shankar, I. Varma, and K. Wilson, *The number of quartic D_4-fields ordered by conductor*, Journal of the European Mathematical Society **23** (2021), Number 8, Pg. 2733-2785

[ATPS20] F. Azpeitia Tellez, C. Powell, and S. Sharif, *Geometry of biquadratic and cyclic cubic log-unit lattices*, preprint, arXiv:1910.12105 (2020).

[BHN25] F. Baril Boudreau, E. Holmes, and K. Nguyen, *Adelic perturbation of rational functions and applications.* Math. Ann. **392**, 2253–2275 (2025).

[Bha04a] M. Bhargava, *Higher composition laws II: On cubic analogues of Gauss composition*, Ann. of Math. (2) **159** (2004), no. 2, 865–886.

[Bha04b] M. Bhargava, *Higher composition laws III: The parametrization of quartic rings*, Ann. of Math. (2) **159** (2004), no. 3, 1329–1360.

[Bha07] M. Bhargava, *Mass formulae for extensions of local fields, and conjectures on the density of number field discriminants*, Int. Math. Res. Not. IMRN (2007), no. 17, rnm052, 20 pp.

[Bha08] M. Bhargava, *Higher composition laws IV: The parametrization of quintic rings*, Ann. of Math. (2) **167** (2008), no. 1, 53–94.

[BH16] M. Bhargava and P. Harron, *The equidistribution of lattice shapes of rings of integers in cubic, quartic, and quintic number fields*, Compos. Math. **152** (2016), no. 6, 1111–1120. MR3518306

[BS14] M. Bhargava and A. Shnidman, *On the number of cubic orders of bounded discriminant having automorphism group C_3, and related problems*, Algebra Number Theory **8** (2014), no. 1, 53–88. MR3207579

[BMS20] W. Bolaños and G. Mantilla-Soler, *The shape of cyclic number fields*, Canadian Mathematical Bulletin. 2023; 66(2):599-609.

[CCH+25] S. R. Zapata Ceballos, S. Chari, E. Holmes, F. Jalalvand, R. Y. Njah Nchiwo, K. O'Connor, F. Ramirez, S. Vemulapalli, *Unit lattices of D4-quartic number fields with signature (2,1)*, To appear in Journal of Number Theory (2025)

[Coh99] H. Cohen, *Advanced Topics in Computational Number Theory*, Graduate Texts in Mathematics, vol. 193, Springer-Verlag, New York, 1999.

[CDDO02] H. Cohen, F. Diaz y Diaz, and M. Olivier, *Enumerating quartic dihedral extensions of* $\mathbf{Q}$, Compos. Math. **133** (2002), 65–93.

[CM11] H. Cohen and A. Morra, *Counting cubic extensions with given quadratic resolvent*, J. Algebra **325** 2011, 461–478. MR2745550

[CT16] H. Cohen and F. Thorne, *On D_ℓ-extensions of odd prime degree ℓ*, Proceedings of the London Mathematical Society 2020 121(5):1171-1206

[Cru20] J. Cruz, *Well-Rounded Unit Lattices from Real Cyclic Number Fields of Small Degree.* M.Sc Thesis, Universidad de los Andes, 2020.

[DGL25] N.-T. Dang, N. Gargava, J. Li, Density of shapes of periodic tori in the cubic case, preprint, arXiv:2502.12754, 2025

[Dav71] H. Davenport and H. Heilbronn, *On the density of discriminants of cubic fields. II*, Proc. Roy. Soc. London Ser. A **322** (1971), no. 1551, 405–420. MR0491593]

[DS18] O. David and U. Shapira, *Dirichlet shapes of unit lattices and escape of mass*, Int. Math. Res. Not. IMRN **2018**, no. 9, 2810–2843. MR3801497

[Fuj23] S. Fujikawa, *The negative Pell's equation in positive characteristic*, Master's thesis, University of Calgary. https://hdl.handle.net/1880/116916

[HH20] P. Harron and R. Harron, *The shapes of Galois quartic fields*, Trans. Amer. Math. Soc. **373** (2020), 7109–7152.

[PH16] P. Harron, *The equidistribution of lattice shapes of rings of integers of cubic, quartic, and quintic number fields: An artist's rendering*, Ph.D. thesis, Princeton University, 2016. MR3487845

[Har17] R. Harron, *The shapes of pure cubic fields*, Proc. Amer. Math. Soc. **145** (2017), no. 2, 509–524. MR3577857

[Har19] R. Harron, *Equidistribution of shapes of complex cubic fields of fixed quadratic resolvent*, Algebra Number Theory **13** (2019), no. 10, 2323–2346.

[HHV26] R. Harron, E. Holmes, and I. Varma, *Shapes of sextic fields and log-terms in Malle's conjecture*, in preparation (2026).

[HHV25] R. Harron, E. Holmes, and S. Vermulapalli, *Unit shapes of D_p fields*, To appear in Mathematische Zeitschrift (2025)

[Has19] J. Hassan, *On shapes of multiquadratic extensions*, Ph.D. thesis, University of Hawaii at Manoa, 2019.

[Hol21] E. Holmes, *Shapes of pure prime degree number fields*, Ph.D. thesis, University of Calgary, ProQuest Dissertations Publishing, 2021.

[Hol22] E. Holmes, *On the shape of pure prime degree number fields*, Journal de Théorie des Nombres de Bordeaux 37 (2025):1-48.

[JS19] A. Jakhar and N. Sangwan, *Integral basis of pure prime degree number fields*, Indian J. Pure Appl. Math. **50** (2019), no. 2, 309–314. DOI:10.1007/s13226-019-0326-7

[Klü05] J. Klüners, *A counterexample to Malle's conjecture on the asymptotics of discriminants*, C. R. Math. Acad. Sci. Paris **340** (2005), no. 6, 411–414.

[Mal02] G. Malle, *On the distribution of Galois groups*, J. Number Theory **92** (2002), 315–322.

[MSM16] G. Mantilla-Soler and M. Monsurrò, *The shape of $\mathbf{Z}/\ell\mathbf{Z}$-number fields*, Ramanujan J. **39** (2016), no. 3, 451–463.

[MSRG20] G. Mantilla-Soler and C. Rivera-Guaca, *A proof of a conjecture on trace-zero forms and shapes of number fields*, Res. Number Theory **6** (2020), 35.

[Mar69] J. Martinet, *Sur l'arithmétique des extensions galoisiennes à groupe de Galois diédral d'ordre* $2p$, Ann. Inst. Fourier (Grenoble) **19** (1969), no. 1, 1–80.

[Min05] H. Minkowski, *Diskontinuitätsbereich für arithmetische Äquivalenz*, J. Reine Angew. Math. **129** (1905), 220–274.

[Mos83] N. Moser, *Théorème de densité de Tchebotareff et monogénéité de modules sur l'algèbre d'un groupe métacyclique*, Acta Arith. **42** (1983), 1–35.

[Ter97] D. Terr, *The distribution of shapes of cubic orders*, Ph.D. thesis, University of California, Berkeley, 1997. MR2697241