Fall 2023 math 135

Some practice problems and general notes/questions for the final

Warning... this is not meant to be a comprehensive study guide. You can/should look at pre-class essentials, past final exams, practice final, the diagnostic quiz, term test, mathematize, lecture notes, etc.

1 Some topics/notes/general questions: Limits, continuity, differentiation, optimization, related rates, differential equations

1.1 Note 1

What does it mean for a limit to exist? Not exist? How do you compute limits? What is the relationship between limits and continuity? What about limits and differentiability? What are the two forms that we usually write when computing the derivative from first principles? Both can be helpful so it is worth knowing both, and/or how they are related.

1.2 Note 2

Be comfortable taking derivatives of known functions (and combinations of said functions) using the rules from class: sum/difference/constant, product/quotient, chain rule. You should also know how to do this for functions which are given algebraically, graphically, verbally, and in a table. Also, you should know how to take the derivative of inverse functions: what is the formula? If you know the derivative of $f(x)$ at $(3, 4)$, you also know the derivative of the inverse at what point? $(\_, \_)$

1.3 Note 3

Know the relationship between continuity and differentiability. How can you use this to create functions that are continuous but not differentiable, or differentiable but not continuous?

1.4 Note 4

What are (horizontal/vertical) asymptotes? How can you find these? Can a function touch/cross its asymptotes? Can a function have 2 different horizontal asymptotes?

1.5 Note 5

Not only do we need to know about computing derivatives we should also know how to interpret the derivative. We have done this a number of times in lecture.

1.6 Note 6

Know different strategies to solve related rates problems. Draw pictures, write down any relevant equations, find the function you need to optimize (this might be an equation of multiple variables), find constraint equations (equations that relate the variables in the function you want to optimize). Write the function you need to optimize as a function of 1 variable, find critical points/endpoints. Which are the mins/maxs?

1.7 Note 7

For differential equations you should be able to match differential equations with provided functions, match slope fields with differential equations, sketch slope fields given differential equations, and sketch solutions of differential equations/initial value problems. You are not expected to be able to solve differential equations in general (this requires integrals/anti deriatives, which we don't know, if it is even possible). Keep in mind that you night be able to write solutions given a slope field (say linear equations)... this happens a few times in the mathematize problems on this section. It is also worth noting that the 'law of natural growth' is the first example of a differential equation that we encountered and it basically says that the quantity grows (or decays) at a rate proportional to its size: $\frac{dy}{dx} = ky$ for some constant k. The functions we studied in this context were exponential functions $y = c\dot{a}^x$ or $y = ce^{kx}$.

1.8 Note 8

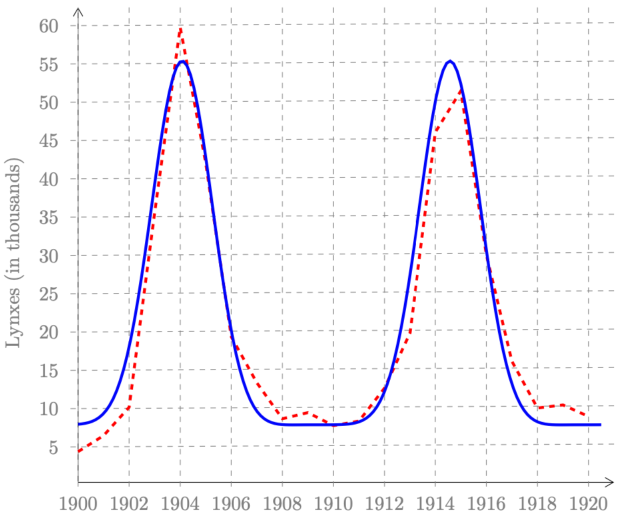
You should know how to approximate functions using linear approximation... how is this related to slope fields? How is this related to Euler's method? If you use a linear approximation (or Euler's method) how do you know when the resulting approximation will be an over/under approximation?

2 Limits/Derivatives

2.1 Lynx problem

Reconsider the lynx problem from the term test. Why does the limit not exist?

8 Consider the graph below with the number of lynxes in Northern Canada between 1900 and 1920. **(14 marks)**



The dashed graph is taken from the actual data collected about the number of lynxes and the solid curve is a **continuous** approximation.

8.1. **(5 marks)** Let us focus on the solid graph of the approximation and assume that the number of lynxes is periodic with a period of 11 years.

Is there a horizontal asymptote? If so, what is it? Explain your reasoning.

(Write DNE in the answer box if there is no horizontal asymptote.)

2.2 Practical interpretation

Let $f(x)$ be the number of meters that Usain Bolt can run in 1 minute when he eats x eggs for breakfast. What is a practical interpretation of $f'(5) = 3$?

2.3 Inverse derivative

Find the derivative of

- $y = \sin^{-1}(x)$.
- $y = f^{-1}(1)$ when $f(x) = 3x^3 + 2x + 1$
- Given the function $g(x)$ approximate $g'(1)$, $g'(3)$, $g''(5)$

x	0	1	2	3	4	5	6
g(x)	-2	1	5	6	9	11	13

  Also compute the derivative of g^{-1} at $y = 5$.
- Find a linear approximation, $L(x)$, for the function $g(x)$ at $x = 6$ and use this to approximate the value of $g(6.5)$ and $g(7)$, Could you use $L(x)$ to approximate values of $g(x)$ that are much larger? Say $x = 100$?

2.4 Question about Gregs talk and log log plots

During a talk in the number theory session of the CMS winter meeting, a mathematician (Greg Knapp) was discussing his recent work. One of his results involves finding a function which bounds a certain data set from above. To figure out the what the function might be he created a log-log plot (pictured in the following figure). Though this is hard to see, the function that bounds the cluster of data (from above) appears to be linear, call this $L(x)$... knowing this, what kind of function do you expect will bound the original data set?

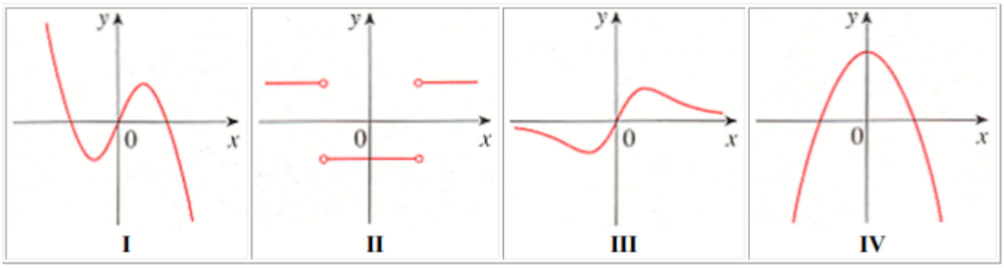
If you know the y-intercept of $L(x)$ is $-.25$ and the x-intercept is $.75$, what is the function that bounds the original data set from above?

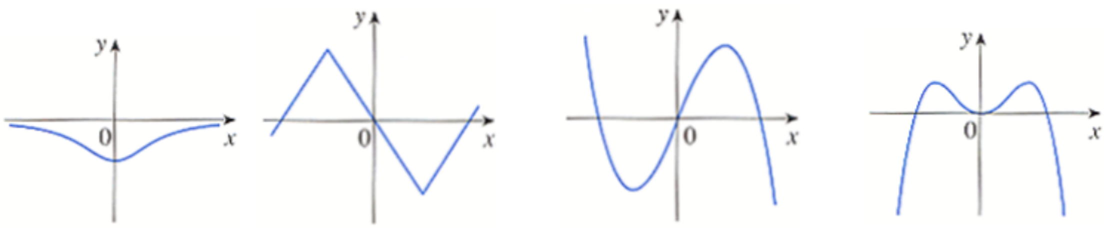
Figure 1: Dr. Gregory Knapp, from the University of Calgary

3 Graphs

3.1 Derivative graphs

Q.1) The graphs of four derivatives are given below. Match the graph of each function in with the graph of its derivative in I-IV.

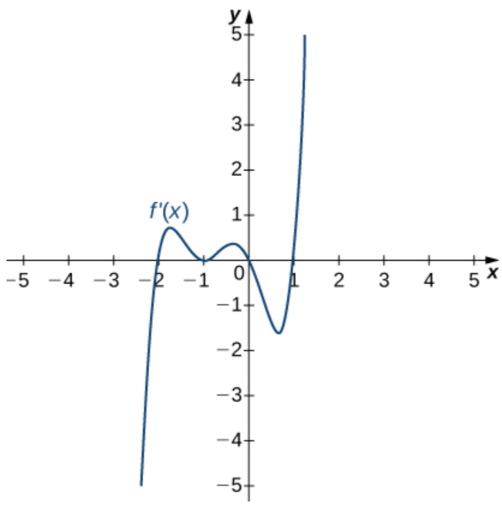




Assuming now that each of the blue graphs is the graph of $f'(x)$ for some $f(x)$, what can you say about critical values, mins/maxs, intervals of increase/decrease?

3.2 Graph 2

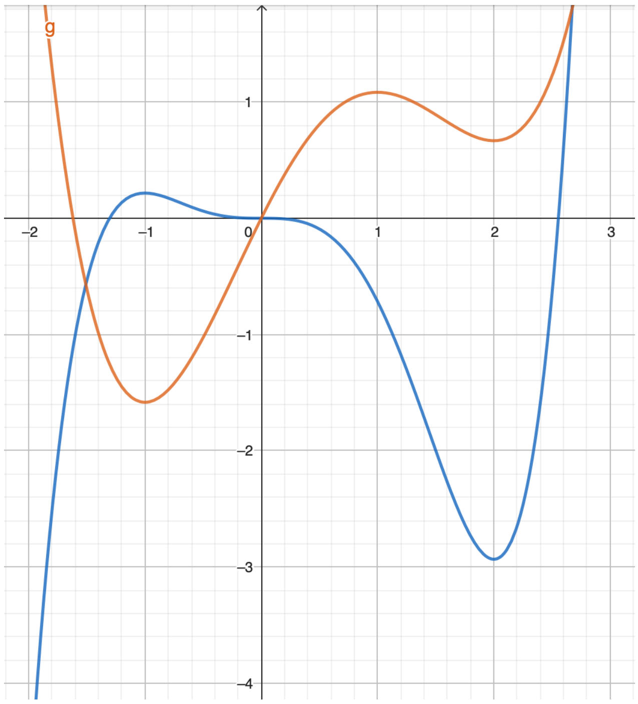
Given the following graph of the derivative, $f'(x)$, sketch the graph of the second derivative. Where are the mins/maxs of the original function, $f(x)$? Where is $f(x)$ increasing/decreasing? Where is $f(x)$ concave up/down? Sketch the graph of the original function $f(x)$.



3.3 Graph 3

Given the following graphs can you determine where the functions below might have local extrema? (warning, this question can be a bit tricky to identify all local extema... the goal is mostly to have you think about derivatives or sum/different/products/compositions and how extrema of these relate to extrema of the original functions)

- $h(x) = f(x)g(x)$
- $k(x) = f(x) + g(x)$
- $m(x) = 3f(x)$
- $n(x) = f(g(x))$

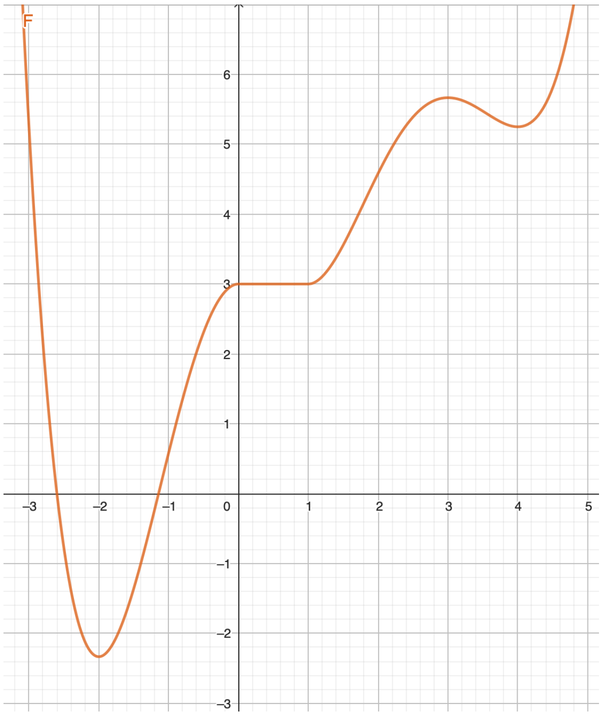


3.4 Graph 4

Given the graph of the following function find the global extrema on the intervals below:

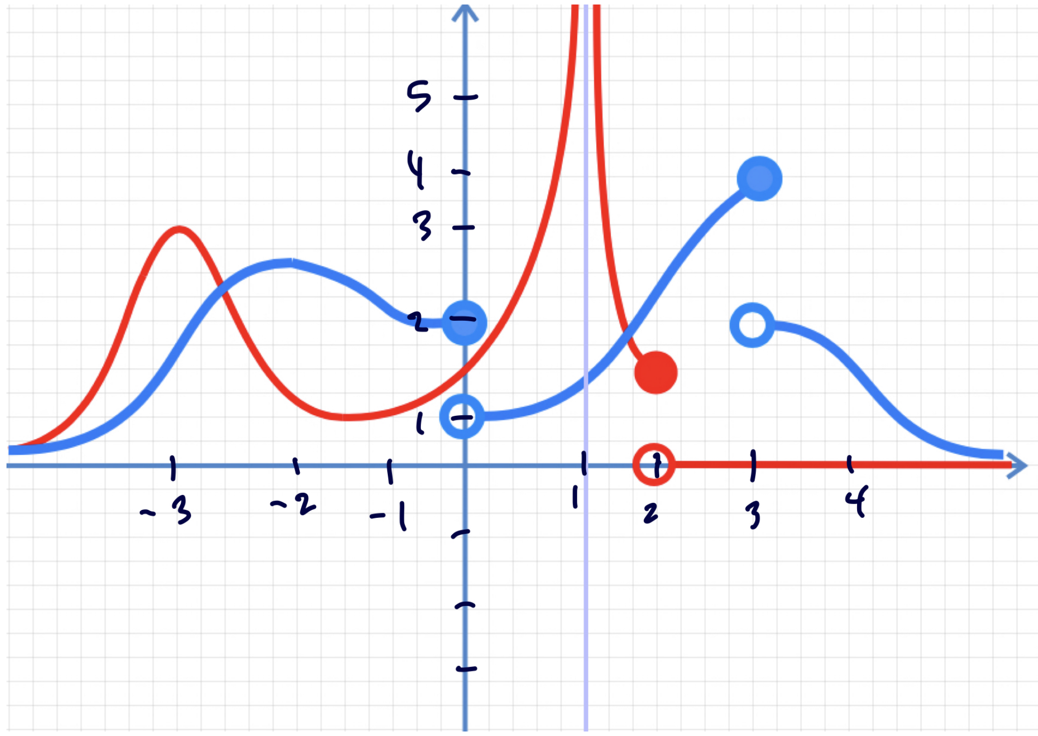
- $[-3, -2]$
- $[-2, 1]$
- $[-3, 2]$
- $[-3, 4]$

Given that the function is continuous, what do you know about the global extrema on these intervals? Are the global extrema also local extrema?



3.5 Graph 5

Given the graph of the curves $f(x)$ and $g(x)$ compute the following limits (if the limit does not exist you can write DNE).



- $\lim_{x \to 1} f(x)$
- $\lim_{x \to 1} f(f(x))$
- $\lim_{x \to -3} g(f(x))$
- $\lim_{x \to -3} f(g(x))$
- $\lim_{x \to \infty} g(f(x))$
- $\lim_{x \to \infty} g(g(x))$

4 Implicit differentiation

4.1 Tilted Ellipse

Find the points, (x, y), on the tilted ellipse, $x^2 - xy + y^2 = 1$, which have the smallest, and largest, values of x. Do the same for y. What is the equation of the tangent line at these points?

4.2 Star

Find all points, (x, y), on the graph $x^{2/3} + y^{2/3} = 8$ for which the tangent line has slope equal to -1.

4.3 More...

Obviously there are more problems out there. Since we have a related rates section though I will keep this one short (remember, you are doing implicit differentiation when solving related rate problems!).

5 Optimization

Below are some optimization/related rates problems that are either made up or pulled from the following site: `https://www.math.ucdavis.edu/~kouba/ProblemsList.html`. You can also check out their calculus page for tips/extra study material: `http://math.ucdavis.edu/~calculus/`

5.1 Leather bag

You are designing a rectangular leather bag that is three times as tall as it is wide. You also want to design a fancy pocket which will be attached to the front of the bag: the pocket is proportional to the dimensions of the main body (suppose the proportion is k = the golden ratio $\approx 13/8$). The only requirements are that the main compartment of the bag needs to be $120cm^3$. Suppose that they plain leather for the main part costs 10 cents per square centimeter, whereas the fancy leather for the pocket costs twice as much. What are the dimensions of the bag that will minimize the cost of building this bag? What is the minimum cost?
What would change if you wanted the entire bag (pocket included) to be $120cm^3$?

5.2 Window design

Construct a window in the shape of a semi-circle over a rectangle. If the distance around the outside of the window is 12 feet, what dimensions will result in the rectangle having largest possible area?

5.3 Orchard

There are 50 apple trees in an orchard. Each tree produces 800 apples. For each additional tree planted in the orchard, the output per tree drops by 10 apples. How many trees should be added to the existing orchard in order to maximize the total output of trees ?

5.4 Movie night

A movie screen on a wall is 20 feet high and 5 feet above the floor. At what distance x from the front of the room should you position yourself so that the viewing angle θ of the movie screen is as large as possible?

5.5 Slope

Of all the lines which are tangent to the graph of $y = \frac{1}{x^2+3}$, find the tangent lines that have the largest and smallest slopes.

6 Related Rates

6.1 Falling box

You are standing 100ft from the base of a 100ft cliff when someone drops a box from the cliff, the box has a parachute attached to it that slows its descent. When the box is released you run towards the base of the cliff. Given that the box is falling at a rate of $4m/s$, and you run at a rate of $3m/s$, at what rate is the distance between you and the box changing when you are halfway to the cliff?
As you run you are tracking the box with your eyes, what rate is the angle between your line of sight and the ground changing at the same instant as above (halfway to the cliff)?

6.2 Boulder

The previous problem was adapted from this one... try it too! You are standing 12 feet from the base of a 200-ft. cliff. As a boulder rolls off the cliff, you begin running away at $10ft/sec$. At what rate is the distance between you and the boulder changing after $t = 1$ and $t = 3$ seconds? (hint: the height of the boulder as a function of time is $s(t) = -16t^2 + 200$).

6.3 Movie night, take two

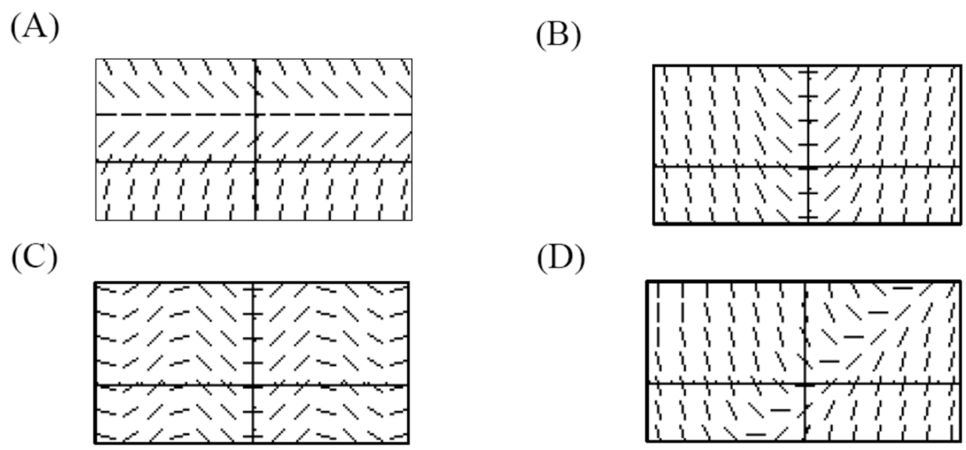
Recall the problem from above (A movie screen on a wall is 20 feet high and 5 feet above the floor...). Suppose you are standing x ft away from the screen: write the viewing angle θ as a function of x. If you are backing away from the screen at a rate of 2ft/sec, at what rate is your viewing angle changing when you are 8 ft from the screen? What about when you are 15ft from the screen?

7 Differential equations

Below are two problems from the following worksheet: `http://mrsk.ca/AP/henricoSlopeFields.pdf`. Check out the link for more problems/study material. *Note: You could also use these to make up some problems that could be useful (like Euler's method with step size 1).*

Match the slope fields with their differential equations. Also, sketch 2 solutions to each differential equation.

7) $dy/dx = \sin x$ **8)** $dy/dx = x - y$ **9)** $dy/dx = 2 - y$ **10)** $dy/dx = x$

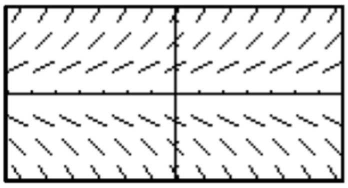


11) $dy/dx = 0.5x - 1$ **12)** $dy/dx = 0.5y$ **13)** $dy/dx = -x/y$ **14)** $dy/dx = x + y$

(A)

(B)

(C)



(D)